Effect of Cross Diffusion on the Onset of Double Diffusive Marangoni Convection in a Fluid Layer

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Abstract

In the present study, double-diffusive convection in a horizontal layer of a fluid in the presence of temperature gradients (Soret effects $\Delta_s$) and concentration gradients (Dufour effects $\Delta_D$) are considered. The onset of convection is studied using linear stability analysis for temperature and concentration boundary conditions. The effects of critical Marangoni numbers, Dufour coefficient $\Delta_D$, Soret coefficient $\Delta_s$ and diffusivity ratio $\tau$ on stationary, oscillatory convection are shown graphically and the effects of various parameters have been discussed in detail.

Keywords: Double Diffusive Convection, Soret Effects, Dufour Effects

Introduction

The convection driven by two different density gradients with differing rates of diffusion is widely known to as “double-diffusive convection” and is an important fluid dynamics phenomenon (See Mojtabi and Charrier-Mojtabi (2005)). The study of double-diffusive convection has attracted attention of many researchers during the recent past due to its occurrence in nature and industry. Oceanography is the root of double-diffusive convection in natural settings. The existence of heat and salt concentrations at different gradients and the fact that they diffuse at different rates lead to spectacular double diffusive instabilities known as “Salt-Fingers” (see Stern (1969)). The formation of salt-fingers can also be observed in laboratory settings. Double-diffusive convection occurs in the sun where temperature and Helium diffusions take place at different rates. Convection in magma chambers and sea-wind formations are among other manifestations of double-diffusive convection in nature. Crystals grown from melts and aqueous solution involve two component systems where the surface-tension depends not only on temperature but also on concentration. Convection generated by such surface-tension forces is called double diffusive Marangoni convection. The study of double-diffusive convection in a fluid/porous medium is of great practical importance in many branches of Science and Engineering, such as Petroleum industry, Food Engineering, Chemical Engineering, Geophysics and Bio-Mechanics. A detailed review is given by Nield & Bejan (2006), with current highly relevant literature including An excellent review of most of the findings related to the above subject has been given by Griffith (1981), Malashetty and Gaikwad (2002), Bhadauria (2007), Nield and Kuznetsov (2007), Kuznetsov and Nield (2010), Gangadharaiah (2013, 2017(a), 2017(b)), Vishwambhar SP, Timol, (2012) and Bhadauria (2011).

McTaggart (1983) has studied the double-diffusive Marangoni convection in a layer bounded below by a rigid adiabatic permeable/impermeable boundary and above by a free boundary with radiation type of condition on temperature and concentration. She has shown that the oscillatory convection does not occur for solute Marangoni number $M_s > 0$ and hence the principle of exchange of stability is valid. However, for $M_s > 0$ oscillatory convection is always preferred to marginal convection. Further, she has found an example of over stability in an aqueous solution of $\text{MgSO}_4$ for which surface–tension increases with concentration. Rudraiah and Siddhehwar (2000) have studied the linear porous medium. They have shown that fingers can be formed by taking the cross diffusion terms.

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of appropriate sign and magnitude even though both components make stabilizing contributions to the net vertical density gradient. They have also shown that finger and diffusive instabilities can never occur simultaneously. The temperature gradient was assumed to vary periodically with respect to time. The two-dimensional instability thresholds, for both oscillatory and stationary instabilities. The theory of double-diffusive convection both theoretically and experimentally is investigated by Turner (1973), Chen et al. (2010), Malashetty and Bharati (2011) and recently Gangadharaiah (2013).

The intent of the present study is to investigate the effect of cross diffusion on the onset of double-diffusive Marangoni convection in a horizontal fluid layer using linear stability analysis for different temperature and concentration boundary conditions. The effect of dimensionless parameters on the onset of double-diffusive Marangoni convection is documented for various forms of basic temperature profiles.

**Physical Models and Mathematical Formulation**

We consider an infinite horizontal layer of an incompressible fluid of depth d subjected to temperature and solute \(\Delta T\) and \(\Delta C\) respectively. The bottom boundary \(z = 0\) is rigid and the upper surface \(z = d\) on which surface tension acts is free and assumed to be non-deformable. Following Pearson [18] the surface tension is assumed to vary linearly with temperature as well as concentration as 
\[
\sigma = \sigma_0 - \sigma_T \Delta T - \sigma_C \Delta C
\]
where \(\sigma_0\) is the unperturbed value and - are the rate of change of surface tension with temperature and concentration respectively.

The governing equations are

\[
\nabla \cdot \mathbf{V} = 0
\]

\[
\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = -\nabla \left( \frac{\rho}{\rho_0} \right) + \gamma \nabla^2 \mathbf{V}
\]

\[
\frac{DT}{Dt} = D_{11} \nabla^2 T + D_{12} \nabla^2 C
\]

\[
\frac{DC}{Dt} = D_{21} \nabla^2 T + D_{22} \nabla^2 C
\]

Where \(\rho = \frac{\partial}{\partial t} \left( \mathbf{V} \cdot \nabla \right)\), \(\mathbf{V}\) is the velocity, \(T\) is the temperature, and \(C\) is the concentration, \(D_{11}\) and \(D_{22}\) are the thermal and solute diffusivities respectively such that while \(D_{12}\) and \(D_{21}\) are the cross-diffusion terms caused by concentration and heat respectively.

The basic state velocity \(\mathbf{V}\), the temperature \(T\) and concentration \(C\) have the solution of the form

\[
\mathbf{V} = 0
\]

\[
\frac{dT}{dz} = -f(z)
\]

\[
\frac{dC}{dz} = -g(z)
\]

Where \(f(z)\) and \(g(z)\) are the basic temperature and concentration gradients respectively, satisfy the condition

\[
\int_0^d f(z) \, dz = -\frac{\Delta T}{d} \quad \text{and} \quad \int_0^d g(z) \, dz = -\frac{\Delta C}{d}
\]

Linearizing the equations using standard perturbation method and making the equations dimensionless using \(d, d^2/\rho_0, D_{11}, D_{22}/\Delta T, \Delta C\) units of the length, time, velocity, temperature and concentration, we obtain (after dropping the primes)

\[
\int_0^1 f(z) \, dz = 1 = \int_0^1 g(z) \, dz
\]

Assuming the solution for \(w, \theta\) and \(C\) in the form

\[
(W, T, C) = [W(z), \theta(z), C(z)] \exp[i(\omega x + my + wt)]
\]

Substituting (14) into (10-12), we obtain

\[
\left[ w - \left( \frac{D^2 - \alpha^2}{\Pr} \right) \right] W = 0
\]

\[
\left[ w - \left( \frac{D^2 - \alpha^2}{\tau} \right) \right] \theta = f(z)W + \Delta_1 \left( \frac{D^2 - \alpha^2}{\Delta C} \right) C
\]

\[
\left[ w - \left( \frac{D^2 - \alpha^2}{\tau} \right) \right] C = \frac{W}{\tau} + \Delta_2 \left( \frac{D^2 - \alpha^2}{\Delta C} \right) \theta
\]

Where \(D = d/dz\), \(\Delta_1 = D_{12}/D_{21} \Delta C/\Delta T\) is the Dufour Coefficient, \(\Delta_2 = D_{22}/D_{11} \Delta T/\Delta C\) is the Soret coefficient, \(\tau = D_{22}/D_{11}\) is the Diffusivity ratio and \(\alpha\) is the square of overall horizontal wave number.

To find the condition for the onset of Marangoni convention, we consider the following boundary conditions

The boundary conditions on velocity are

\[
W = 0 = DW \quad \text{at} \quad z = 0
\]

\[
W = D^2 \theta + M_\alpha \alpha^2 \theta + M_\beta \alpha^2 C \quad \text{at} \quad z = 1
\]

The boundary conditions on temperature are

\[
D\theta = 0
\]

On the insulating boundary.

The boundary condition on solute concentration on

\[
DC = 0
\]

On the permeable boundary.

**Solution by Gelerkin Method**

To obtain the solution to the eigenvalue problem, a single-term Gelerkin expansion technique is employed as it gives very good approximation to the problem considered with minimum mathematical computations. Accordingly \(W, \theta\) and \(C\) in (15) – (17) are replaced by \(AW(z), B\theta(z)\) and \(EC(z)\), where \(A, B,\) and \(E\) are constant amplitudes. Then these equations are respectively multiplied by \(W, \theta\) and \(C\) and integrated by parts from \(z=0\) to \(1\) to obtain the following equations.
\( (wG + F) + M_1 HB + M_s IE = 0 \) \hspace{1cm} (18)
\( NA + (wJ + M) B + KE = 0 \) \hspace{1cm} (19)
\( RA + LB + (Q - wP) E = 0 \) \hspace{1cm} (20)

Where
\[ G = \left( (DW)^2 + a^2W^2 \right), \]
\[ F = \left( (D^2W)^2 + 2a^2(DW)^2 + a^4W^2 \right), \]
\[ H = a^2DW(1) \theta(1), \quad J(\theta^2) \]
\[ N = (f(z)W)M = (D\theta^2 + a^2\theta^2), \]
\[ K = \Delta_1(D\theta DC + a^2\theta C), P = (C^2), R = (g(z)W)C \]
\[ L = \Delta_1(D\theta DC + a^2\theta C), \quad Q = (DC)^2 + a^2C^2 \]

The relevant trial functions chosen satisfying the boundary conditions are
\[ W = z^2(1 - z^2) \]
\[ \theta = 3z^2 - 2z^3 \text{(both boundaries insulating)} \]
\[ C = 3z^2 - 2z^3 \text{(both boundaries impermeable)} \]

**Marginal Stationary State**

In this section, we assume that the marginal state is valid and set \( w = 0 \). Expression for Marangoni number for different temperature profile is

1. **Linear Temperature \( f(z) = 1 \)**

\[ M_r = \frac{b_0(b_0 - \Delta_1b_1)}{a^2[23rb_1(420) - \Delta_1b_1]} + M_s \frac{C(1)(b_0 - (23rb_1(420)))}{\theta (1) \left(23rb_1(420) + b_1\right)} \hspace{1cm} (21) \]

2. **Parabolic Temperature \( f(z) = 2z \)**

\[ M_r = \frac{b_0(b_0 - \Delta_1b_1)}{a^2[63rb_1(840) - \Delta_1b_1]} + M_s \frac{C(1)(b_0 - (61rb_1(840)))}{\theta (1) \left[61rb_1(840) + b_1\right]} \hspace{1cm} (22) \]

3. **Inverted parabolic temperature \( f(z) = 2(1-z) \)**

\[ M_r = \frac{b_0(b_0 - \Delta_1b_1)}{a^2[31rb_1(840) - \Delta_1b_1]} + M_s \frac{C(1)(b_0 - (31rb_1(840)))}{\theta (1) \left[31rb_1(840) + b_1\right]} \hspace{1cm} (23) \]

Where
\[ b_1 = \frac{1 + 2a^2}{15}, \quad b_2 = \frac{23b_1}{420}, \quad b_3 = \frac{84}{5} + \frac{88a^2}{105} + \frac{8a^4}{315}, \]
\[ b_4 = \frac{6 + 13a^2}{5}, \quad b_5 = \frac{37b_4}{420} \]

**Marginal Oscillatory State**

Oscillatory instability can occur provided the constraining mechanisms work in opposition. That is, when one of the gradients induce a stabilizing effect and the other a destabilizing effect. In other words, when one mechanism cause motion to amplify and the other suppresses it then convection sets in as over stable motion. McTaggart (1983), as shown that the oscillatory instability is possible in the case of concentration driven by surface tension due to variation of temperature and concentration, only when \( M_s < 0 \).

In this section, we investigate the possibility of occurring oscillatory instability, when the cross diffusion effects are present for a representative boundary condition. Expression for Marangoni number is

\[ \frac{\delta_0 \delta_1\left(\delta_0 \delta_2 - \delta_1 \delta_1\right) + M_s \left(\delta_0 \left(fW\theta\right) - \delta_1 \left(gW\theta\right)\right)}{a^2 \theta (1)DW (1)} \]

Putting \( w = iv \) and separating real and imaginary parts, we obtain

\[ M_r = b_0 + b_2w^2 + b_4w^4 + M_s \frac{b_{10} - b_{11}w^2}{b_6} \]

\[ M_r = b_0 + b_2w^2 + b_4w^4 + M_s \frac{b_{10} - b_{11}w^2}{b_6} \]

Since \( M_r \) is real number, \( \text{Im}(M_r) = 0 \), we get

\[ \lambda_1 w^4 + \lambda_2 w^2 + \lambda_3 = 0 \]

Where
\[ b_6 = \frac{1 + 23a^2}{15}, b_7 = \frac{23rb_1(1 + rb_1)}{420}, b_8 = \frac{37b_2}{420} + \frac{103b_2}{440}, b_9 = \frac{37b_1}{420}, b_{10} = \frac{46}{63000}, b_{11} = \frac{rb_1(23 + rz b_1)}{420}, b_{12} = \frac{23}{63000}(5 + 2a^2) \]

**Exact solutions for the steady case \( w = 0 \)**

We seek a closed form solution for the marginal stability curve of the steady \( w = 0 \) Marangoni convection and by setting \( w = 0 \) in (11), solution of \( w \) (2) which satisfies the boundary condition (16), we get

\[ W = \frac{1}{2} \left((1 + \delta_1 z) \sinh (az) - az \cosh (az)\right) \]

Substituting (27) in to (12)-(13) and using boundary conditions

\[ D\theta (0) = DC (0) = DC (1) = D\theta (1) = 0 \]

\[ \theta = A \left\{ k_1 - k_2z \frac{z - k_1}{2a} \right\} \sinh az + \left\{ \frac{k_2}{2a} + \frac{k_1}{2a} \right\} \frac{z}{4a} \]

\[ C = A \left\{ k_0 - k_4 \frac{z - k_5}{4a} z \right\} \sinh az + \left\{ \frac{k_5}{2a} + \frac{k_4}{2a} z - \frac{k_5}{4a} z \right\} \cosh az \]
\[
\frac{k_5}{4a^2z^2} \cosh az
\]  

Finally expression for Marangoni number is
\[
M_f = \frac{k_5 - M_1a^2}{a^2} \left\{ k_1 \cosh a + k_3 \sinh a \right\} (30)
\]

Where
\[
k_1 = \left( \frac{1}{4} - \frac{k_2}{4a^2} + \frac{k_4}{2a^2} \right) \cosh a, \quad k_3 = -k_5 \left( \frac{1}{4} + \frac{k_2}{4a^2} + \frac{k_4}{4a^2} \right),
\]
\[
k_2 = \frac{1}{\tau} - \Delta_1 k_1, \quad k_3 = \frac{1}{\tau} - \Delta_2 k_2
\]
\[
k_4 = -\frac{a}{\Delta_1} \left( \frac{1}{4} + \frac{k_2}{4a^2} + \tau k_4 k_2 \right)
\]
\[
k_5 = -k_5 \left( \frac{1}{2a} - \frac{k_2}{4a^2} + \frac{k_4}{4a^2} \right), \quad k_8 = \left( \frac{k_1}{4a^2} + \frac{k_4}{4a^2} \right)
\]
\[
k_6 = -a^2 \left\{ k_1 a^2 \cosh a + 2ak_8 \sinh a \right\}
\]

Results and Discussion

The effect of different forms of non-uniform basic temperature gradient. On the onset of double diffusive Marangoni convection with cross diffusion effects is investigated. Both marginal and over stable states are studied. The resulting Eigen value problem is also solved exactly for steady case.

For the better understanding of liner stability criteria, it is important to study the topology of neutral curves. Figures 1-7 show the curves in the \(M_f - \alpha\). Figure 1 is a graph for solute Marangoni number \(M_f = -50, 50\) and \(\tau = 0.32\) in the absence of the cross diffusion effects (i.e. \(\Delta_1 = 0, \Delta_2 = 0\)) . From this figure we note that the curve are connected in a topological sense, and the linear stability criteria can be expressed in term of a critical Marangoni number below which the system is stable and above which it is definitely unstable. For \(M_f = -50\) The curve for \(w^2 = 0\) lies above the curve for \(w^2 > 0\) but the opposite behavior could be seen for \(M_f = 50\) in other words, for \(M_f = 50\) oscillatory convection is ruled out. However, in the presence of cross diffusion effect (i.e. \(\Delta_1 = 0.1, \Delta_2 = 1.0\)) an exact opposite situation may be noticed (see Fig. 2-4). Thus the presence of cross diffusion terms completely alters the dynamics of the system.

Figure 5-7 is a graph for \(M_f = -50, 50\) for \(\Delta_1 = 0.1\) and \(\Delta_2 = 0\), in the presence of only Soret effect (i.e. without Dufour effect), it is observed that both stationary and oscillatory neutral curves are connecting at two bifurcation points and the curve for \(w^2 = 0\) lies below the curve for \(w^2 > 0\) . Nevertheless, the presence of both Dufour and Soret effects make the system to behave in an opposite way. That is curve for \(w^2 > 0\) lies below the curve for \(w^2 = 0\), and hence the instability sets in as oscillatory motions.

The effects of diffusivity ratio, \(\tau\) on the stability of the system are depicted in Fig. 5-7. Figure 6 is a graph for \(\tau = 0.6\). It is clearly seen from the figure that in the absence of cross diffusion effect convection sets in as steady type while an opposite behavior may be noticed when cross diffusion effects are present. It may be noted that the two bifurcation points comes closer as compared to that of \(\tau = 0.32\). As it is increased further from 0.6 to 0.8 of cross diffusion effects (see Fig. 7), whereas in the presence of cross diffusion effects these two curves connect at two points (see Fig. 3)

The critical Marangoni numbers obtained for different values of temperature profiles are exhibited in Table 1, 2 and 3. We note the linear temperature profile is the most unstable profile in the absence of cross diffusion terms and also in the presence of only Dufour effect. However, in the presence of both the cross diffusion terms it is noted that the linear temperature profile is the most stable one. In order to know the accuracy of the results obtained from the

![Figure 1](image1.png)

**Figure 1.** Marangoni number versus wave number for \(\Delta_1 = 0 = \Delta_2\) and \(\tau = 0.32\).

![Figure 2](image2.png)

**Figure 2.** Marangoni number versus wave number for \(\Delta_1 = 0.1, \Delta_2 = 1.0\) and \(\tau = 0.32\).
Figure 3. Marangoni number versus wave number for $\Delta_1 = 0.1, \Delta_2 = 1.0$ and $\tau = 0.6$

Figure 4. Marangoni number versus wave number for $\Delta_1 = 0.1, \Delta_2 = 1.0$ and $\tau = 0.8$

Figure 5. Marangoni number versus wave number for $\Delta_1 = 0.1, \Delta_2 = 0$ and $\tau = 0.32$

Figure 6. Marangoni number versus wave number for $\Delta_1 = 0.1, \Delta_2 = 0$ and $\tau = 0.6$

Figure 7. Marangoni number versus wave number for $\Delta_1 = 0.1, \Delta_2 = 0$ and $\tau = 0.8$

Table 1. The Marangoni number $M_f$ for different values of $M_s$ with different temperature Profiles for $\Delta_1 = 0, \Delta_2 = 0$

<table>
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<tr>
<th>$M_s$</th>
<th>$f(z) = 1$</th>
<th>$f(z) = 2z$</th>
<th>$f(z) = 2(1-z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-50.0</td>
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<td>164.526</td>
<td>44.926</td>
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<tr>
<td>-40.0</td>
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<td>-12.361</td>
</tr>
<tr>
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<td>-68.58</td>
<td>-18.726</td>
</tr>
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Table 2. The Marangoni number $M_f$ for different values of $M_s$ with different temperature profiles for $\Delta_1 = 0.1, \Delta_2 = 0$

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Galerkin technique the problem is also solved exactly for the steady case. The results obtained by both the methods are found to be in good agreement and same is evidenced from the Tables 4 for case of linear temperature profile.

**Conclusion**

In this study we used linear stability analysis to investigate cross-diffusion (Soret and Dufour) effects in double-diffusive (thermal and solutal gradients imposed) convection in a horizontal fluid layer. The aim of this work was to investigate the Soret and Dufour effects on the onset of convection. It was found that, in the case of stationary instability, the Soret effect had a stabilizing effect whereas the Dufour effect was destabilizing. The cross diffusion effects were found to have no effect on stability.

**References**


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