Overview Studies on Numerical Methods for Gas Dynamics

Pramod V, Abhishek Shukla

Abstract

Numerical Methods plays very crucial role as for as gas dynamics is concern. This work performs the numerical convergence studies of different numerical methods used for gas dynamics over a range of parameters and analyze the accuracy of the approach applied to gas dynamics.

Keywords: **Numerical Methods, gas dynamics, Euler equations, shock-capturing methods, Flux -Vector Splitting Schemes, HLL and HLLC Riemann solvers.**

Euler Equations

In this section we consider the time-dependent Euler equations. These are a system of non-linear hyperbolic conservation laws that govern the dynamics of a compressible material, such as gases or liquids at high pressures, for which the effects of body forces: viscous stresses and heat flux are neglected. There is some freedom in choosing a set of variables to describe the flow under consideration. A possible choice is the so called *primitive variables* or *physical variables,* namely, *p(x,* y, *z, t)* = density or mass density, *p(x.* y, *z ,t)* = pressure, *u(x,* y , *z , t)* = 2-component of velocity, *v(x ,*y , *z . t)* =y-component of velocity, *w (z ,* y, *z , t)* = z-component of velocity. The velocity vector is V = *(u, v, w)*.A n alternative choice is provided by the so called *conserved variables.* These are the mass density *p:* the z-momentum component *pu,* the y-momentum component *pv.* the z-momentum component *pu,* and the total energy per unit mass E. Physically, these conserved quantities result naturally from the application of the fundamental laws of conservation of mass, Yen-ton's Second Lam and the law of conservation of energy. Computationally, there are some advantages in expressing the governing equations in terms of the conserved variables.

Conservation-law form

The five governing conservation laws are:

$$\rho_t + (\rho u)_x + (\rho u)_y + (\rho u)_z = 0 \quad (1)$$

$$(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y + (\rho uw)_z = 0 \quad (2)$$

$$(\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y + (\rho vw)_z = 0 \quad (3)$$

$$(\rho w)_t + (\rho uw)_x + (\rho vw)_y + (\rho w^2 + p)_z = 0 \quad (4)$$

$$E_t + [u(E+p)]_x + [v(E+p)]_y + [w(E+p)]_z = 0 \quad (5)$$

E is the total energy per unit volume

$$E = \rho(\frac{1}{2}\mathrm{V}^2 + e) \quad (6)$$

Where $\frac{1}{2}\mathrm{V}^2 = \frac{1}{2}\mathrm{V}.\mathrm{V} = \frac{1}{2}(u^2 + v^2 + w^2)$

½ V^2 is the *specific kinetic energy* and *e* is the *specific internal energy.* One generally refers to the full system as the Euler equations. The conservation laws can be expressed in a very compact notation by defining a column vector U of conserved variables and flux vectors **F(U)**, **G(U)**, **H(U)** in the *x, y* and *z* directions, respectively. The equations now read

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x + \mathbf{G}(\mathbf{U})_y + \mathbf{H}(\mathbf{U})_z = 0 \quad (7)$$

With

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho\, u \\ \rho\, v \\ \rho\, w \\ E \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \rho\, u \\ \rho\, u^2 + p \\ \rho\, u\, v \\ \rho\, u\, w \\ u\,(E + p) \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \rho\, v \\ \rho\, u\, v \\ \rho\, v^2 + p \\ \rho\, v\, w \\ v\,(E + p) \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} \rho\, w \\ \rho\, u\, w \\ \rho\, v\, w \\ \rho\, w^2 + p \\ w\,(E + p) \end{bmatrix} \tag{8}$$

It is important to note that F = F(U), G = G(U), H = H(U); that is, the flux vectors are to be regarded as functions of the conserved variable vector U. Any set of PDEs written in the form is called a system of *conservation laws*. As partial derivatives are involved we say that is a system of conservation laws in differential form. The differential formulation assumes smooth solutions, that is, partial derivatives are assumed to exist. There are other ways of expressing conservation laws in which the smoothness assumption is relaxed to include discontinuous solutions.

One dimensional Euler equations in conservative formulation

The conservative formulation of the Euler equations, in differential form, is

$$\mathbf{U_t} + \mathbf{F}(\mathbf{U})_x = 0 \tag{9}$$

Here $\mathbf{U}$ and $\mathbf{F}(\mathbf{U})$ are the vectors of conservative variables and fluxes given respectively by

$$\mathbf{U} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \equiv \begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}, F = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \equiv \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(E+p) \end{bmatrix} \tag{10}$$

Here ρ is density, p is pressure, u is particle velocity, E is Total energy per unit volume

$$E = \rho\left(\frac{1}{2}u^2 + e\right) \tag{11}$$

Where e is the *specific internal energy* given by a caloric equation of state (EOS)

$$e = e(\rho, p) \tag{12}$$

For ideal gases one has simple expression

$$e = e(\rho, p) = \frac{p}{(\gamma - 1)\rho} \tag{13}$$

With $\gamma = c_p / c_v$ denoting the ratio of specific heats from EOS (13) and now we write the sound speed a as

$$a = \sqrt{p/\rho^2 - e_\rho / e_p} = \sqrt{\frac{\gamma p}{\rho}} \tag{14}$$

The conservation laws can be written in quasi linear form

$$\mathbf{U_t} + \mathbf{A}(\mathbf{U})\mathbf{U}_x = 0 \tag{15}$$

Where the coefficient matrix $\mathbf{A}(\mathbf{U})$ is the *Jacobian matrix*

$$\mathbf{A}(\mathbf{U}) = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} = \begin{bmatrix} \partial f_1/\partial u_1 & \partial f_1/\partial u_2 & \partial f_1/\partial u_3 \\ \partial f_2/\partial u_1 & \partial f_2/\partial u_2 & \partial f_2/\partial u_3 \\ \partial f_3/\partial u_1 & \partial f_3/\partial u_2 & \partial f_3/\partial u_3 \end{bmatrix} \tag{16}$$

$$\mathbf{A}(\mathbf{U}) = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2}(\gamma-3)(\frac{u_2}{u_1})^2 & (3-\gamma)(\frac{u_2}{u_1}) & \gamma-1 \\ -\frac{\gamma u_2 u_3}{u_1^2} + (\gamma-1)(\frac{u_2}{u_1})^3 & \frac{\gamma u_3}{u_1} - \frac{3}{2}(\gamma-1)(\frac{u_2}{u_1})^2 & \gamma\frac{u_2}{u_1} \end{bmatrix} \quad (17)$$

First we express all components f_i of the flux vector **F** in terms of the components *u,* of the vector **U** of conserved variables, namely $u_1=\rho$, $u_2=\rho u$, $u_3= E$. Obviously $f_1 = u_2=\rho u$. To find f2 and f3 we first need to express the pressure *p* in terms of the conserved variables. From (11) and (13) we find

$$p = (\gamma-1)[u_3 - \frac{1}{2}(u_2^2/u_1)]$$

The flux vector can be written as:

$$\mathbf{F}(\mathbf{U}) = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \equiv \begin{bmatrix} u_2 \\ \frac{1}{2}(3-\gamma)\frac{u_2^2}{u_1} + (\gamma-1)u_3 \\ \gamma\frac{u_2}{u_1}u_3 - \frac{1}{2}(\gamma-1)\frac{u_2^3}{u_1^2} \end{bmatrix}$$

By direct evaluation of all partial derivatives we arrive at sought result

Now we write the jacobian matrix in terms of the sound speed a and velocity *u*

$$\mathbf{A}(\mathbf{U}) = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma-3)(u)^2 & (3-\gamma)(u) & \gamma-1 \\ \frac{1}{2}(\gamma-2)(u)^3 - \frac{a^2 u}{\gamma-1} & \frac{3-2\gamma}{2}(u)^2 + \frac{a^2}{\gamma-1} & \gamma u \end{bmatrix} \quad (18)$$

The Euler equations (9)-(10) with the ideal-gas EOS (13) satisfy the homogeneity property

$$\mathbf{F(U) = A(U)U} \quad (19)$$

The proof of this property is immediate. By multiplying the Jacobian matrix (18) by the vector **U** in (10) we identically reproduce the vector **F(U)** of fluxes in (10).

The Eigen values of the jacobian matrix **A** are obtained from the expression (18) for A and the characteristic polynomial

$$|A - \lambda I| = 0 \text{ Lead to}$$

$$(\lambda-u)(\gamma u-\lambda)[2u-\gamma u-\lambda)] + (\lambda-u)[-a^2-(\gamma-1)u^2+(\gamma-1)\gamma u^2] + \Delta$$

Where

$$\Delta = \frac{1}{2}(\gamma u-\lambda)(1-\gamma)u^2 - \frac{1}{2}(\gamma-1)u^2[(1-2\gamma)\lambda+\gamma u]$$

Manipulations show that Δ also contains the common factor $(\lambda - u)$, which implies that $\lambda_2 = u$ is a root of the characteristic polynomial and thus an eigen values of A. After cancelling $\lambda - u$ the remaining terms give

$$\lambda^2 - 2u\lambda + u^2 - a^2 = 0$$

With real roots

$$\lambda_1 = u - a, \lambda_2 = u, \lambda_3 = u + a$$

Therefore the eigen values are: $\lambda_1 = u - a, \lambda_2 = u, \lambda_3 = u + a$ as claimed to find the right eigenvectors for a vector $K = [k_1, k_2, k_3]^T$ such that

$$\mathbf{AK} = \lambda \mathbf{K}$$

Substituting $\lambda=\lambda_{\ I}$ in turn, solving for the components of vector K and selecting appropriate values for the scaling factors we get the desired eigenvectors.

The eigen values are all real and the eigenvectors $K^{(1)}$, $K^{(2)}$, $K^{(3)}$forma complete set of linearly independent eigenvectors. We have thus proved that the time-dependent, one-dimensional Euler equations for ideal gases are hyperbolic. In fact these equations are strictly hyperbolic, because the eigen values are all real and distinct, as long as the sound speed a remains positive.

Shock Capturing Methods

In computational fluid dynamics, shock-capturing methods are a class of techniques for computing in viscid flows with shock waves. Computation of flow through shock waves is an extremely difficult task because such flows result in sharp, discontinuous changes in flow variables pressure, temperature, density, and velocity across the shock.

In shock-capturing approach the governing equations of in viscid flows (Euler equations) are cast in conservation form and any shock waves or discontinuities are computed as part of the solution. Here, no special treatment is employed to take care of the shocks themselves. This is in contrast to the shock-fitting method, where shock waves are explicitly introduced in the solution using appropriate shock relations (Rankine - Hugoniot relations).

The shock capturing methods are relatively simple compared to the more elaborate shock fitting methods. However, the shock waves predicted by shock-capturing methods are generally not sharp and smear over several grid points. Also, classical shock-capturing methods have the disadvantages that unphysical oscillations (Gibbs phenomenon) may develop in the vicinity of strong shocks.

Classical and modern shock capturing methods

From an historical point of view, shock-capturing methods can be classified into two general categories: viz., *classical methods* and *modern shock capturing methods* (also called high-resolution schemes). Modern shock-capturing methods are generally upwind based in contrast to classical symmetric or central discretization. Upwind-type differencing schemes attempt to discretize hyperbolic partial differential equations by using differencing biased in the direction determined by the sign of the characteristic speeds. On the other hand, symmetric or central schemes do not consider any information about the wave propagation in the discretization.

No matter what type of shock-capturing scheme is used, a stable calculation in presence of shock waves requires a certain amount of numerical dissipation, in order to avoid the formation of unphysical numerical oscillations. In the case of classical shock-capturing methods, numerical dissipation terms are usually linear and the same amount is uniformly applied at all grid points. Classical shock-capturing methods only exhibit accurate results in the case of smooth and weak-shock solution, but when strong shock waves are present in the solution, non-linear instabilities and oscillations can arise across discontinuities. Modern shock-capturing methods have, however, a non-linear numerical dissipation, with an automatic feedback mechanism which adjusts the amount of dissipation in any cell of the mesh, in accord to the gradients in the solution. These schemes have proven to be stable and accurate even for problems containing strong shock waves.

Some of the well-known classical shock-capturing methods include the MacCormack method (uses a discretization scheme for the numerical solution of hyperbolic partial differential equations), Lax–Wendroff method (based on finite differences, uses a numerical method for the solution of hyperbolic partial differential equations), and Beam-Warming method. Examples of modern shock-capturing schemes include, higher order Total Variation Diminishing (TVD) schemes first

proposed by Harten, Flux-Corrected Transport scheme introduced by Boris and Book, Monotonic Upstream-centered Schemes for Conservation Laws (MUSCL) based on Godunov approach and introduced by Van Leer, various Essentially Non-Oscillatory schemes (ENO) proposed by Harten et al., and Piecewise Parabolic Method (PPM) proposed by Woodward and Colella. Another important class of high resolution schemes belongs to the approximate Riemann solvers proposed by Roe and by Osher. The schemes proposed by Jameson and Baker, where linear numerical dissipation terms depend on non-linear switch functions, fall in between the classical and modern shock-capturing methods.

Flux -Vector Splitting Schemes (FVS)

Classical shock-capturing methods use central differences. The underlying idea behind flux-vector splitting is to split the flux contributions into positive and negative components, where splitting is based on the eigen value structure of the system or some other appropriately assumed behavior. In presenting these methods, the view is taken that the fundamental problem that must be solved is to determine the correct flux at the boundaries of the control-volume faces. Interpretation of the numerical methods in terms of the control-volume surface fluxes for the various methods may also be considered in the sense of finite-difference schemes.

To set the stage for the study of solutions of the Euler equation, consider a control volume as shown in Fig. 1. As previously discussed, the conservative form of the governing equations is integrated over the control volume. The 2-D Euler equations are given in the conservative form:

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{E}}{\partial x}+\frac{\partial \mathbf{F}}{\partial y}=0 \qquad (20)$$

where the conservative variables are defined in the usual way. Integrating this equation over the control volume yields the form.

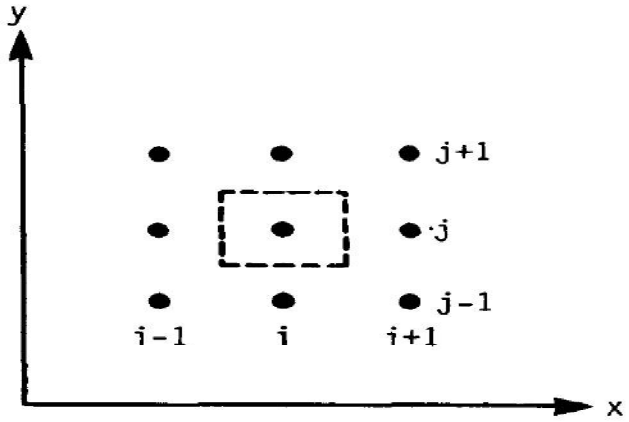

Figure 1.Control volume for Euler equations

$$\int_{\delta v}\frac{\partial \mathrm{U}}{\partial \mathrm{t}}dv+\int_{\delta v}(\frac{\partial \mathrm{E}}{\partial x}+\frac{\partial \mathrm{F}}{\partial y})dv=0 \qquad (21)$$

Applying Green's Theorem (Taylor, 1955) to the second term converts this to a surface integral of the form

$$\int_{\delta v}\frac{\partial \mathrm{U}}{\partial \mathrm{t}}dv+\oint_{s}(\mathrm{E}\,dy-\mathrm{F}\,dx)=0 \qquad (22)$$

where the subscript on the integral around the boundary is denoted by the small s. In discrete form, the integration results in

$$\frac{\partial \mathrm{U}}{\partial \mathrm{t}}\partial v+\sum_{cellfaces}(\mathrm{E}\,\Delta \mathrm{y}-\mathrm{F}\,\Delta x)=0 \qquad (23)$$

where the δv represents the volume of the cell and the Δx and Δy are the arc lengths of the cell sides for the **2-D** case. The evaluation of the sum of the fluxes on the boundary requires that the flux values, i.e., the values of **E** and **F,** be known on the surface of the control volume. The evaluation of the flux terms on the control volume surfaces is the fundamental problem in the development of methods for solving the Euler equations.

Steger –Warming Flux Vector Splitting

Steger and Warming (1979) developed an implicit algorithm using a splitting of **E** and **F** in the governing equations based on the wave speed splitting. In splitting the flux terms, the flux is assumed to be composed of a positive and a negative component. For illustration, consider a **1-D** problem where the Euler system under investigation has the form

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{E}}{\partial x}=0 \qquad (24)$$

This system can also be written in the form

$$\frac{\partial \mathbf{U}}{\partial t}+[A]\frac{\partial \mathbf{U}}{\partial x}=0 \qquad (25)$$

Where *[A]*is the Jacobian $\partial \mathbf{E}/\partial \mathbf{U}$ this system is hyperbolic if a similarity transformation exists so that

$$[T]^{-1}[A][T]=[\lambda] \qquad (26)$$

where [λ] is a diagonal matrix of real Eigen values of *[A]*and $[T]^{-1}$ is the matrix whose rows are the left eigenvectors of *[A]*taken in order.

$$p=\rho f(e) \qquad (27)$$

Where ***e*** is the internal energy, then the flux vector **E(U)** is a homogeneous function of degree one in **U,** which means that

$$\mathbf{E}(\alpha \mathbf{U})=\alpha \mathbf{E}(\mathbf{U}) \qquad (28)$$

for any α. This permits the flux vectors **E** and **F** of the Euler equations to be written in the form

$$\mathbf{E}=[A]\mathbf{U} \qquad (29)$$

We can use this property and the fact that the system is hyperbolic to achieve the desired split flux form.

Combining equations (26) and (29) **E** may be written as

$$\mathbf{E}=[A]\mathbf{U}=[T][\lambda][T]^{-1}\mathbf{U} \qquad (30)$$

The matrix of Eigen values is divided into two matrices, one with only positive elements and the other with negative elements. We write the ***[A]*** matrix as

$$[A]=[A^{+}]+[A^{-}]=[T][\lambda^{+}][T]^{-1}+[T][\lambda^{-}][T]^{-1} \qquad (31)$$

And define

$$\mathbf{E}=\mathbf{E}^{+}+\mathbf{E}^{-} \qquad (32)$$

So that $\mathbf{E}^{+}=[A^{+}]\mathbf{U}$ $\mathbf{E}^{-}=[A^{-}]\mathbf{U}$ (33)

The original conservation-law form written using the split-flux notation becomes

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{E}^{+}}{\partial x}+\frac{\partial \mathbf{E}^{-}}{\partial x}=0 \qquad (34)$$

where the plus and minus signs indicate that the flux components are associated with wave propagation in the positive and negative directions, respectively. The key point is that the flux vector **E** can be split into a positive part and a negative part, each associated with the signal propagation directions. The eigen values of $\partial \mathbf{E}^{\pm}/\partial \mathbf{U}$ are not the same as $\lambda^{\pm}$, but the correct sign is preserved. For the *1-D* case, the eigen values of [A] are the familiar streamline and signal propagation terms written as

$$\lambda_1=u$$
$$\lambda_2=u+a$$
$$\lambda_3=u-a$$

For the supersonic case, with *u* positive, $\lambda^{+}=\lambda$ and $\lambda^{-}=0$. For the subsonic case, both λ^{+} and λ^{-} are nonzero. For subsonic flow

$$[\lambda^{+}]=\begin{bmatrix} u & & \\ & u+a & \\ & & 0 \end{bmatrix} \qquad (35)$$

$$[\lambda^{-}]=\begin{bmatrix} 0 & & \\ & 0 & \\ & & u-a \end{bmatrix} \qquad (36)$$

The associated split-flux terms are as follows:

$$\mathbf{E}^{-}=\frac{1}{2}\frac{\rho}{\gamma}(u-a)\begin{bmatrix}1\\ u-a\\ \frac{1}{2}(u-a)^2+\frac{1}{2}a^2\left(\frac{3-\gamma}{\gamma-1}\right)\end{bmatrix} \quad (37)$$

$$\mathbf{E}^{+}=\mathbf{E}-\mathbf{E}^{-}=\frac{1}{2}\frac{\rho}{\gamma}\begin{bmatrix}(2\gamma-1)\\ 2(\gamma-1)u^2+(u+a)^2\\ (\gamma-1)u^3+\frac{1}{2}(u+a)^3+\frac{1}{2}a^2\frac{3-\gamma}{\gamma-1}(u+a)\end{bmatrix} \quad (38)$$

A first-order upwind scheme is easily constructed with this split-flux idea. A simple integration of the equations for a *1-D* problem may be written

$$\mathbf{U}_i^{n+1}=\mathbf{U}_i^{n}-\frac{\Delta t}{\Delta x}(\mathbf{E}_{i+1/2}-\mathbf{E}_{i-1/2}) \quad (39)$$

In this setting, the cell-face values of the flux are composed of both + and -components according to the splitting, i.e.

$$\mathbf{E}_{i+1/2}=(\mathbf{E}^{+}+\mathbf{E}^{-})_{i+1/2} \quad (40)$$

For a first-order calculation the flux components may be evaluated with an extrapolation consistent for the MUSCL scheme, where the primitive variables were extrapolated to the cell faces. In the Steger-Warming splitting, the fluxes are extrapolated to the cell faces. However, the MUSCL approach with primitive variables may also be used in this splitting. In the simplest case, the values of $\mathbf{E}_{i+1/2}^{+}$ are set equal to $\mathbf{E}_{i}^{+}$, and the values of $\mathbf{E}_{i+1/2}^{-}$ are set equal to $\mathbf{E}_{i+1}^{-}$ This produces a numerical algorithm of the form

$$\mathbf{U}_i^{n+1}=\mathbf{U}_i^{n}-\frac{\Delta t}{\Delta x}[(\mathbf{E}^{+}+\mathbf{E}^{-})_{i+1/2}-(\mathbf{E}^{+}+\mathbf{E}^{-})_{i-1/2}] \quad (41)$$

This is the finite-difference form of Eq. (41) when the **E+** derivative is backward differenced and the **E-** term is forward differenced. Based on earlier discussions, the equivalence of the finite-difference and the finite-volume formulations is clear.

The use of split-flux techniques for shock-capturing applications produces better results than central-difference methods, but some problems remain even for this formulation. Using the Steger-Warming splitting, the shock waves are well represented, but some oscillations are produced when a sonic condition is encountered. The problem is that the components of the split flux are not continuously differentiable at sonic and stagnation points. Figure 2 shows the split mass flux behavior as the sonic region is traversed. Steger and Warming (1981) attempted to eliminate this problem by modifying the eigen values when they change signs to be of the form

$$\lambda^{\pm}=\frac{\lambda\pm\sqrt{\lambda^2+\varepsilon^2}}{2} \quad (42)$$

where **ε** is viewed as a blending function to ensure a smooth transition when the **λ's** change sign. This modification was only moderately successful and more appropriate schemes employing flux-vector splitting evolved later.

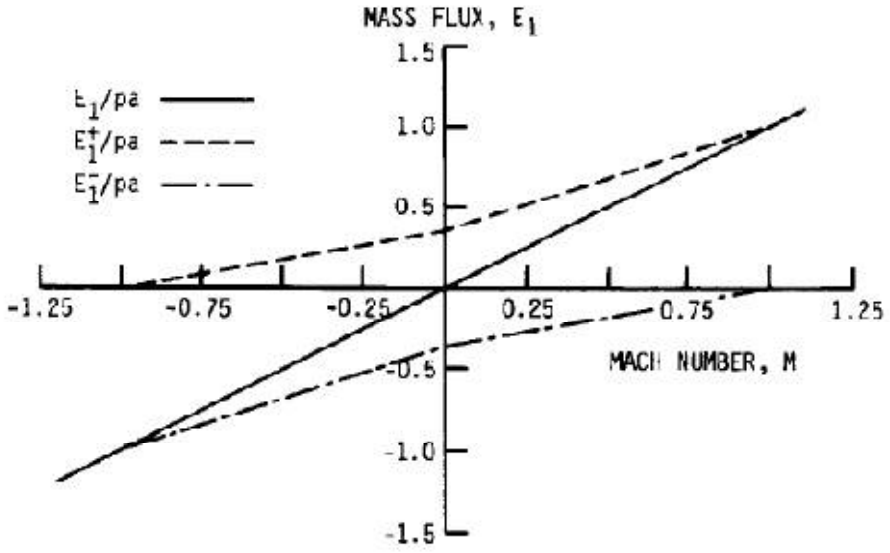

Figure 2.Split mass flux using steger-warming splitting

Van Leer Flux Vector splitting

Van Leer (1982) suggested a second type of flux vector splitting, Unlike the Steger- Warming flux vector splitting. Van Leer's flux vector splitting is not based on wave speed splitting sonic points are natural flux splitting points; at the very least sonic points require special consideration to avoid numerical problems. For the Euler equations, the Mach number M indicates sonic points; in particular, sonic points occur when M = u/a =0 ± 1 thus to help address sonic points. Van Leer bases his Flux Vector splitting on Mach number splitting. Van Leer flux splitting takes a logical approach to the problem of flux calculation. The flux vector is analyzed to determine its eigenvectors, which are the wave speeds of the equation set for the Euler equations. For the quasi 1-D equation set, there are three speeds: [$u+a$, u, and $u-a$]. The flux is split into the respective contributions from each wave speed, with a "positive" or "left" flux component coming from the positive wave speeds and a "negative" or "right" flux component coming from the negative wave speeds. If the flow is supersonic, only the flux component from the upstream direction is non-zero. For subsonic flow, both the left and right flux components are non-zero. The left flux is calculated from conservative variables interpolated using a left biased stencil, while the right flux is calculated from conservative variables interpolated using a right biased stencil. In this way, each flux component is correctly calculated from known cell points which would affect the boundary.

The flux vector for the Euler equations can be written in terms of the Mach number as follows:

$$\mathrm{f} = \begin{bmatrix} \mathrm{f}_1 \\ \mathrm{f}_2 \\ \mathrm{f}_3 \end{bmatrix} = \begin{bmatrix} \rho a M \\ \rho a^2(\gamma M^2 + 1) \\ \rho a^3 M(\frac{1}{2}M^2 + \frac{1}{\gamma - 1}) \end{bmatrix} \qquad (43)$$

Notice that the mass flux f_1 Depends linearly on M, then the mass flux can be split much like the flux function for the linear advection equation In particular the linear function M can be split into two quadratic functions as follows:

$$M^+ = \begin{cases} 0 & M \le -1 \\ (\frac{M+1}{2})^2 & -1 < M < 1 \\ M & M \ge 1 \end{cases} \qquad (44)$$

$$M^{-}=\begin{cases} M & M\le -1 \\ -(\frac{M+1}{2})^2 & -1<M<1 \\ 0 & M\ge 1 \end{cases} \qquad (45)$$

Notice that $M^+ + M^- = M$ for $|M| \ge 1$ the same true for $|M| < 1$ which is easily proven as follows

$$M^{+}+M^{-}=\frac{M^2+2M+1}{4}-\frac{M^2-2M+1}{4}=M$$

Notice that $M^{\pm}$ and its first derivative are continuous. Unfortunately, the second derivative is highly discontinuous at sonic points $M=\pm 1$. This could be prevented by splitting the linear flux into two cubic pieces-the higher the order polynomials in the splitting the more degrees of freedom there are ensure continuity in the derivatives. Mach number splitting (45) implies the following mass splitting.

$$f_1^{\pm}=\rho a M^{\pm} \qquad (46)$$

The momentum flux f_2 depends on γM^2+1. By the same principles as before; this quadratic is split into two cubics where the cubics ensure continuity of split momentum of the split momentum flux and its derivative. Omitting the details the result is:

$$(\gamma M^2+1)^{+}=\begin{cases} 0 & M\le -1 \\ (\frac{M+1}{2})^2((\gamma-1)M+2) & -1<M<1 \\ \gamma M^2+1 & M\ge 1 \end{cases} \qquad (47)$$

$$(\gamma M^2+1)^{-}=\begin{cases} 0 & M\le -1 \\ -(\frac{M+1}{2})^2((\gamma-1)M+2) & -1<M<1 \\ \gamma M^2+1 & M\ge 1 \end{cases} \qquad (48)$$

This simplifies the following momentum flux splitting:

$$f_2^{\pm}=\frac{\rho a^2}{\gamma}(\gamma M^2+1)^{\pm} \qquad (49)$$

$$f_2^{+}=\begin{cases} 0 & M\le -1 \\ \frac{1}{\gamma}f_1^{+}((\gamma-1)u+2a) & -1<M<1 \\ f_2 & M>1 \end{cases} \qquad (50)$$

$$f_2^{-}=\begin{cases} f_2 & M\le -1 \\ \frac{1}{\gamma}f_1^{-}((\gamma-1)u-2a) & -1<M<1 \\ 0 & M>1 \end{cases} \qquad (51)$$

Momentum flux includes a pressure term, by standard convention; then the above momentum flux splitting implicitly involves a pressure splitting. Van Leer splits the pressure as follows:

$$p^{+} = p\begin{cases} 0 & M \le -1 \\ (\frac{M+1}{2})^2(2-M) & -1 < M < 1 \\ 1 & M \ge 1 \end{cases} \quad (52)$$

$$p^{-} = p\begin{cases} 1 & M \le -1 \\ -(\frac{M+1}{2})^2(2-M) & -1 < M < 1 \\ 0 & M \ge 1 \end{cases} \quad (53)$$

Finally, the energy flux depends on the cubic $M(\frac{1}{2}M^2 + \frac{1}{\gamma-1})$, which is split into two quadratics.

$$f_3^{+} = \begin{cases} 0 & M \le -1 \\ \frac{1}{2(\gamma+1)(\gamma-1)} f_1^{+}((\gamma-1)u+2a)^2 & -1 < M < 1 \\ f_3 & M > 1 \end{cases} \quad (54)$$

$$f_3^{-} = \begin{cases} f_3 & M \le -1 \\ \frac{1}{2(\gamma+1)(\gamma-1)} f_1^{-}((\gamma-1)u-2a)^2 & -1 < M < 1 \\ 0 & M > 1 \end{cases} \quad (55)$$

To summarize from discussed equations, notice that

$$f^{\pm} = \pm\frac{\rho a}{4}(M \pm 1)^2 \begin{bmatrix} 1 \\ \frac{(\gamma-1)u \pm 2a}{\gamma} \\ \frac{((\gamma-1)u \pm 2a)^2}{2(\gamma+1)(\gamma-1)} \end{bmatrix} \quad (56)$$

For |M|<1 other wise, f^{+} =f and f^{-} =0 for $M \ge 1$, and f^{-} =f and f^{+} = 0 for $M \le 1$ Like the Steger-Warming flux vector splitting, Van Leer flux vector splitting correctly attributes all of the flux to right running waves for left-running supersonic flow.

HLL and HLLC Riemann solvers

Computing the Godunov flux, Harten, Lax and van Leer presented a novel approach for solving the Riemann problem approximately. The resulting Riemann solvers have become known as HLL Riemann solvers. In this approach an *approximation for the intercell numerical flux is obtained directly*. The central idea is to assume a wave configuration for the solution that consists of two waves separating three constant states. Assuming that the wave speeds are given by some algorithm, application of the integral form of the conservation laws gives a closed-form, approximate expression for the flux. The approach produced practical schemes after the contributions of Davis and Einfeldt who independently proposed various ways of computing the wave speeds required to completely determine the intercell flux.

The resulting HLL Riemann solvers form the bases of very efficient and robust approximate Godunov-type methods. One difficulty with these schemes, however, is the assumption of a two-wave configuration. This is correct only for hyperbolic systems of two equations, such as the one-dimensional shallow water equations. For larger systems, such as the Euler equations or the split two-dimensional shallow water equations for example, the two-wave assumption is incorrect. As a consequence the resolution of physical features such as contact surfaces, shear waves and material interfaces, can be very inaccurate. For the limiting case in which these features are stationary relative to the mesh, the resulting numerical smearing is unacceptable. In view of these shortcomings of the HLL approach, a modification called the HLLC Riemann solver (C stands for Contact) was put forward by Toro, Spruce and Speares. In spite of the limited experience available in using the HLLC scheme, the evidence is that this appears to offer a useful approximate Riemann solver for practical applications. Batten, Leschziner and Goldberg have recently proposed implicit versions of the HLLC Riemann solver, and have applied the scheme to turbulent flows.

References

1. Milcev SS, Pavlovic MD. Influence of Spike Shape at Supersonic Flow Past Blunt-Nosed Bodies.
2. University of Belgrade, 11000 Belgrade, Serbia, Yugoslavia publication, 29 January 2002. Copyright 2002 by the American Institute of Aeronautics and Astronautics, Inc.
3. Nagdewe SP, Vaghela MB, Shevare GR. Comparison of turbulence models for the Simulation of Internal and External flows. IIT Bombay, India. Proceedings of the XXVII Iberian Latin American Congress on Computational methods in Engineering, 2006.
4. Toro EF. Riemann Solvers and Numerical methods for Fluid Dynamics: A practical Introduction. Second edition *Springer*, 1999.
5. Spalart P, Allarmas S. A one equation turbulent model for Aerodynamic flows. AIAA paper 92-0439, 1992.
6. Morgenstern Jr A. Three-dimensional supersonic flow over a spike-nosed body of revolution. *Journal of the Brazilian Society of Mechanical Sciences* Nov 2002; 24(4).
7. Woodward P, Collella P. Numerical Simulation of 2Dimensional fluid flow with Strong shocks. *Journal of Computational Physics* 1984; 54: 115-73.
8. Clain S, Rouchette D, Touzani R. A Multi Slope MUSCL Method for Unstructured Meshes Applied to Compressible Euler Equations for Swirling Flows, 19 Nov 2008.
9. Anderson JD. Modern compressible flow By John D. 1990.
10. Collella‘A P. Direct Eulerian MUSCL Scheme for Gas Dynamics ’SIAM *J. Sci. Stat.* January 1985; 6.
11. Blazek J. Computational Fluid Dynamics: Principles and Applications, 2001.
12. Laney CB. Computational gas Dyanmics.